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The Banach Approximation Problem

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The problem alluded to in the title is whether every (real or complex Banach space E has the approximation property, i.e., the property that the identity operator on E can be approximated uniformly on each compact subset of E by bounded linear operators of finite-dimensional range. Recently Enflo [2] solved this problem in the negative; a simplified version of his construction can be found in [1]. The object of this paper is to show how the construction in [1] can be framed in terms of some concrete formulations of the problem due to Grothendieck [3].

Grothendieck showed that the following 3 assertions are equivalent:

(1) every Banach space has the approximation property;

(2) if $A = (a_{ij} : i, j = 1, 2,...)$ is an infinite matrix satisfying $\sum_i \sup_j | a_{ij} | < \infty$ and $A^2 = 0$ then trace $(A) = 0$.

(3) if f is continuous on the unit square $[0, 1] \times [0, 1]$ and

$$\int_0^1 f(x, t)\, f(t, y)\, dt = 0 \qquad \text{for all} \quad x, y \in [0, 1]$$

then

$$\int_0^1 f(t, t)\, dt = 0.$$

(in (2) and (3) a_{ij} and f may be assumed to be either real-valued or complex-valued).

A COUNTEREXAMPLE TO (2)

We construct a matrix A which disproves (2) as follows: for $k = 0, 1, 2,...$ let U_k be a unitary matrix of order 3. 2^k Partition U as

$$U = \begin{bmatrix} 2^{k+1/2} P_k \\ 2^{k/2} Q_k \end{bmatrix}$$

where P_k has 2^{k+1} rows and Q_k has 2^k rows. Put

$$A = \begin{bmatrix} P_0{}^*P_0 & P_0{}^*Q_1 & 0 & 0 & 0 & \cdot \\ -Q_1{}^*P_0 & P_1{}^*P_1 - Q_1{}^*Q_1 & P_1{}^*Q_2 & 0 & 0 & \cdot \\ 0 & -Q_2{}^*P_1 & P_2{}^*P_2 - Q_2{}^*Q_2 & P_2{}^*Q_3 & 0 & \cdot \\ 0 & 0 & -Q_3{}^*P_2 & P_3{}^*P_3 - Q_3{}^*Q_3 & P_3{}^*Q_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

where the rows and columns are grouped in blocks of 3, 6, 12, 24,.... It is easily checked that $A^2 = 0$, using the relations

$$P_kP_k{}^* = 2^{-k-1}I_{2^{k+1}}, \qquad Q_kQ_k{}^* = 2^{-k}I_{2^k}, \qquad P_kQ_k{}^* = 0, \qquad Q_kP_k{}^* = 0$$

which follow from U_k being unitary. Moreover the trace of each diagonal block is zero, except the first. Hence to disprove (2) it suffices to show that the U_k can be chosen so that $\sum_i \sup_j | a_{ij} | < \infty$.

In fact we can choose U_k so that each element of the kth block of rows is bounded by $C(k)^{1/2}\, 2^{-3k/2}(*)$, C being a constant. Since the kth block contains 3.2^k rows this implies $\sum_i \sup_j | a_{ij} | \leqslant \sum_{k=0}^{\infty} 3C(k)^{1/2}\, 2^{-k/2} < \infty$ as required. Indeed we have $\sum_i \sup_j | a_{ij} |^p < \infty$ whenever $p > \frac{2}{3}$.

The U_k are constructed by putting on Abelian group structure on $\{1, 2,..., 3.2^k\}$ (e.g., as a cyclic group), splitting the set of characters on this group into two sets $\{\tau_i : i = 1,..., 2^{k+1}\}$ and $\{\sigma_i : i = 1,..., 2^k\}$, and letting the rows of P_k be $3^{1/2}\, 2^{-(2k+1)/2}\tau_i$ and the rows of Q_k be $3^{1/2}\, 2^{-k}\epsilon_i\sigma_i$ where $\epsilon_i = \pm 1$. By a probabilistic argument one shows that for "most" choices of the splitting of characters and of the numbers ϵ_i ($i = 1,..., 2^k$), the estimate (*) holds. Details may be found in [1].

A Counterexample to (1)

Following Grothendieck we can use the matrix A above to construct a space without the approximation property. Let a_i be the ith row of A; we regard a_i as an element of the Banach space c_0 of sequences converging to zero with the supremum norm. Let E be the closed linear span of $\{a_i\}$ in c_0. Then E does not have the approximation property. To prove this we define a linear functional ϕ on the space $B(E)$ of all bounded linear operators on E by $\phi(T) = \sum_i T(a_i)_i$. Then $| \phi(T)| \leqslant (\sum_i i^{-5/4}) \sup_i \| T(i^{5/4}a_i)\|$ so ϕ is continuous w.r.t. the topology on $B(E)$ of uniform convergence on the compact set $\{i^{5/4}a_i\} \cup \{0\}$. If $S(x) = x_j l_k$ then $\phi(S) = \sum_i a_{ki}a_{ij} = 0$—since every operator of finite rank is in the closed linear span of such operators S—it follows that $\phi(T) = 0$ for all finite rank T. But $\phi(I) = \text{trace}\ (A) \neq 0$, which completes the proof.

A similar argument shows that we cannot get a counterexample to (2) satisfying $\sum_i \sup_j |a_{ij}|^{2/3} < \infty$.

Suppose we could. Let $\lambda_i = \sup_j |a_{ij}|$; we may assume $\lambda_i > 0$ for all i. Let $b_{ij} = \lambda_i^{-1/3}\lambda_j^{1/3}a_{ij}$ and let B be the matrix (b_{ij}). Then $B^2 = 0$, trace $(B) =$ trace $(A) \neq 0$, and $\sum_i (\sum_j |b_{ij}|^2)^{1/2} < \infty$.

Then we can argue as above with l^2 in place of C_0, regarding the rows of B as element of l^2, and get a subspace of l^2, not having the approximation property, which is impossible.

A Counterexample to (3)

Again following Grothendieck we can use the matrix A constructed above to find a function f disproving (3). Let $\rho_i = (10i)^{-1}(1 + \log i)^{-2}$. Since $\sum \rho_i < 1$ we can find a sequence of disjoint intervals I_i on [0, 1] with $|I_i| = \rho_i$. Let ϕ_i be a continuous function vanishing outside I_i with $\int \phi_i^2 = \rho_i$ and $|\phi_i(x)| \leqslant 2$, $|\phi_i'(x)| \leqslant 8\rho_i^{-1}$. Put $f(x, y) = \sum_{i,j} a_{ij}\rho_i^{-1}\phi_i(x)\,\phi_j(y)$.

It is easily checked that f has the desired properties. Indeed f satisfies a Lipschitz condition of order α for each $\alpha < \frac{1}{2}$.

References

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